Intersection theory on surfaces

We fix terminology for the rest of the course:

surface : nonsingular projective variety $/ k = \bar{k}$ of dimension 2

curves: effective divisors on a surfaces (might be irreducible or unreduced)

pts : closed points of a surface

The intersection pairing

X surface

D_1, D_2 effective divisors

cut out by sections $s_i \in H^0(X, \mathcal{O}_X(D_i)) \setminus \{0\}$

$$0 \to \mathcal{O}_X(-D_i) \xrightarrow{s_i} \mathcal{O}_X \to \mathcal{O}_{D_i} \to 0$$

intersection $D_1 \cap D_2$ cut out by (s_1, s_2)

$\Rightarrow \forall x \in X : \quad \mathcal{O}_{D_1 \cap D_2, x} = \mathcal{O}_{X,x} / (f_1, f_2) \qquad s_i|_x = f_i$

if D_1, D_2 don't share an irreducible component, then $D_1 \cap D_2$ is finite dimensional

<u>Def</u> $D_1, D_2 \subset X$ effective divisors s.t. $\dim(D_1 \cap D_2) = 0$

<u>intersection multiplicity</u> @ $x \in D_1 \cap D_2$

$$\mathrm{mult}_x(D_1 \cap D_2) = \dim_k(\mathcal{O}_{D_1 \cap D_2, x})$$

<u>intersection number</u>

$$(D_1 . D_2) = h^0(X, \mathcal{O}_{C \cap D}) = \sum_{x \in D_1 \cap D_2} \dim_k(\mathcal{O}_{D_1 \cap D_2, x})$$

<u>Lemma</u> $D_1, D_2 \subset X$ effective divisors s.t. they don't share a component.

Then $\dim(\mathcal{O}_{D_1 \cap D_2, x}) = 0$ for all $x \in X$

<u>Pf</u> X smth $\Rightarrow$ factorial.

$D_1, D_2 \subset X$ don't share a component

$\Rightarrow$ local equations f_1, f_2 don't have a common factor

$\Rightarrow$ $(f_1, f_2) \not\subseteq (g)$ $\forall g \in \mathfrak{m}_{X,x}$

$\Rightarrow$ $\dim(\mathcal{O}_{D_1 \cap D_2, x}) = 0$ $\square$

<u>Lemma</u> The sequence

$$0 \to \mathcal{O}_X(-D_1 - D_2) \xrightarrow[(s_2\ \ -s_1)]{} \mathcal{O}_X(-D_1) \oplus \mathcal{O}_X(-D_2) \xrightarrow{\binom{s_1}{s_2}} \mathcal{O}_X \to \mathcal{O}_{D_1 \cap D_2} \to 0$$

is exact.

<u>Proof</u> local question.

w.t.s. $0 \to \mathcal{O}_{X,x} \xrightarrow{(f_2\ -f_1)} \mathcal{O}_{X,x}^{\oplus 2} \xrightarrow{\binom{f_1}{f_2}} \mathcal{O}_{X,x} \to \mathcal{O}_{D_1 \cap D_2, x} \to 0$ is exact.

only exactness at $\mathcal{O}_{X,x}^{\oplus 2}$ not immediate.

$(a,b) \in \ker\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \Rightarrow -f_1 a = f_2 b$

D_1, D_2 don't share irred component $\Rightarrow \gcd(f_1, f_2) = 1$

$\Rightarrow -f_1 \mid b \ , \ f_2 \mid a$ and $c := \frac{a}{f_2} = \frac{b}{-f_1}$

$\Rightarrow \ a = f_2 c \ , b = -f_1 c \ \Rightarrow (a,b) = \mathrm{im}\,(f_2 \ -f_1)$ $\square$

<u>Cor</u> $H^0(\mathcal{O}_{D_1 \cap D_2}) = \chi(\mathcal{O}_X) - \chi(\mathcal{O}_X(-C)) - \chi(\mathcal{O}_X(-D))$
$+ \chi(\mathcal{O}_X(-C-D))$

<u>Lemma</u> $C \subset X$ is irreducible curve.
$D \subset X$ (effective) divisor.

$$\Rightarrow \quad (C.D) = \deg_C(\mathcal{O}_C(D)|_C)$$

<u>Pf</u> have s.e.s.

$$0 \to \mathcal{O}_X(-C) \to \mathcal{O}_X \to \mathcal{O}_C \to 0$$

$$0 \to \mathcal{O}_X(-D-C) \to \mathcal{O}_X(-D) \to \mathcal{O}_X(-D)|_C \to 0$$

$\Rightarrow \chi(\mathcal{O}_C) = \chi(\mathcal{O}_X) - \chi(\mathcal{O}_X(-C))$

$\chi(\mathcal{O}_X(-D)|_C) = \chi(\mathcal{O}_X(-D)|_C) - \chi(\mathcal{O}_X(-D-C))$

$\Rightarrow (C.D) = \chi(\mathcal{O}_C) - \chi(\mathcal{O}_X(-D)|_C)$

$\underset{\substack{RR \\ \text{for curves}}}{=} 1-g - (1-g) + \deg_C(\mathcal{O}_X(-D)|_C)$

$= \deg(\mathcal{O}_X(D)|_C)$ $\square$

Theorem $(\,.\,): Cl(X)\times Cl(X)$

$(D_1.D_2) = \chi(\mathcal{O}_X) - \chi(\mathcal{O}_X(-D_1)) - \chi(\mathcal{O}_X(-D_2)) - \chi(\mathcal{O}_X(-D_1-D_2))$

is a symmetric bilinear pairing.

Proof for $C \subset X$ nonsingular irred.

$(C.\,D_1+D_2) = \deg_C(\mathcal{O}_X(D_1+D_2)|_C)$

$= \deg(\mathcal{O}_X(D_1)|_C) + \deg(\mathcal{O}_X(D_2)|_C)$

$= (C.D_1) + (C.D_2)$

want to reduce to this case

Fix $\mathcal{L}$ very ample line bundle on X

Bertini $\Rightarrow$ $\mathcal{L} = \pi^* \mathcal{O}_X(1) \cong \mathcal{O}_X(H)$

for $H \subset X$ ns. irreducible.

$\forall\ D_0, D_1, D_2 \in Cl(X)$ w.t.s.

$s(D_0, D_1, D_2) := (D_0.D_1) + (D_0.D_2) - (D_0.\,D_1+D_2) = 0$

Pick $n>0$ s.t. $\mathcal{O}(D_i + nH)$ v. ample.

$\ \ \cong$

Bertini $\Rightarrow$ $\mathcal{O}(C_i)$, $\mathcal{O}_X(nH) \cong \mathcal{O}_X(B)$

C_i, B ns. proj. irred.

by Euler characteristic definition

$\Rightarrow s(D_0, D_1, D_2) = s(D_{\sigma 0}, D_{\sigma 1}, D_{\sigma 2})\ \forall \sigma \in S_3$

$\Rightarrow s(D', C_0, -B) = s(C_0, D', -B) = 0$

$\Rightarrow (D'.\underbrace{C_0 - B}_{=D}) = (D', C) + (D', -B)$

$\Rightarrow (D_0.\,D_1+D_2) = (C.\,D_1+D_2) - (B.\,D_1+D_2)$

$$= (C.D_1)+(C.D_2)-(B.D_1)-(B.D_2)$$

$$= (D_0.D_1)+(D_0.D_2)$$

used also

$s(D', B, -B) = 0 \Rightarrow 0 = (D.B) + (D.-B)$ $\square$

Example. $X \cong \mathbb{P}^2$, $\mathrm{Pic}(\mathbb{P}^2) = \mathbb{Z}\ell$

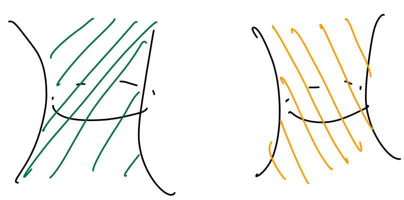
$\ell^2 = \ell\,\ell' =$ $\quad$ ℓ, ℓ' $\forall$ lines

$D_1, D_2 \subset \mathbb{P}^2$ of deg $d_1 d_2$

$\Rightarrow (D_1.D_2) = d_1 d_2$

Example $X \cong \mathbb{P}^1 \times \mathbb{P}^1$ $\quad \mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1) = \mathbb{Z}\ell_1 \oplus \mathbb{Z}\ell_2$

$\ell_1^2 = 0 = \ell_2^2 \qquad \ell_1.\ell_2 = 1$



if D_1 of type (a_1, a_2)

D_2 of type (b_1, b_2)

then $(D_1.D_2) = a_1 b_2 + a_2 b_1$

__Theorem__ (Riemann-Roch)

$K = K_X$ canonical divisor D divisor

$$\chi(\mathcal{O}_X(D)) = \frac{1}{2}(D.D-K) + \chi(\mathcal{O}_X)$$
$$= \frac{1}{2}(D.D-K) + \chi(\mathcal{O}_X)$$

__Proof__

$$(-D.D-K) = \chi(\mathcal{O}_X) - \chi(\mathcal{O}_X(D)) - \chi(\mathcal{O}_X(K-D)) + \chi(\mathcal{O}_X(K))$$
$$\underset{\text{Serre duality}}{=} 2\chi(\mathcal{O}_X) - 2\chi(\mathcal{O}_X(D)) \qquad \square$$

__Remark__ H (very) ample $(D.H) \overset{\text{Exercises}}{=} \deg_H(D) > 0$

__Lemma__ H ample divisor on X

$\exists n_0 \in \mathbb{Z}$ $\forall D$ divisor on X

$$(D.H) > n_0 \Rightarrow H^2(X, \mathcal{O}_X(D)) = 0$$

__Pf__ $h^2(\mathcal{O}_X(D)) = h^0(\mathcal{O}_X(K-D))$

$h^0(\mathcal{O}_X(K-D)) \neq 0 \Rightarrow K-D$ effective.

$\Rightarrow (K-D.H) > 0 \Rightarrow (K.H) > (D.H)$

Take $n_0 = (K.H)$ $\square$

__Cor__ H ample divisor on X, D divisor on X

$D.H > 0$ and $D^2 > 0 \Rightarrow nD \sim_{lin}$ effective for $n \gg 0$

Pf $\quad D.H > 0 \quad \Rightarrow \quad nD.H > n_0 \quad$ for $n \gg 0$

$$\Rightarrow \quad h^2(nD) = 0 \quad \text{for } n \gg 0$$

$$h^0(nD) \overset{RR}{=} \frac{1}{2}(nD.nD - K) + \chi(\mathcal{O}_X) + h^1(nD)$$

$$\geq \frac{1}{2} n^2 D^2 - \frac{1}{2} nD.K + \chi(\mathcal{O}_X)$$

$$\xrightarrow[n \to \infty]{} \infty \qquad \Rightarrow nD \text{ effective } n \gg 0 \qquad \square$$

Def

$$Cl(X) \longrightarrow Hom(Cl(X), \mathbb{Z}) = Cl(X)^\vee$$

$$D \longmapsto (D.-)$$

$D_1 \sim_{num} D_2$ iff $(D_1.-) = (D_2.-)$

Theorem (Hodge index theorem)

H ample divisor on X, D divisor on X

$$\left.\begin{array}{l} D \not\sim_{num} 0 \\ (D.H) = 0 \end{array}\right\} \Rightarrow D^2 < 0$$

Pf Suppose $D \not\sim_{num} 0 \quad (D.H) = 0$.

Step 1 Suppose $D^2 > 0$. $\quad H' := D + nH$ ample for $n \gg 0$

$$(D.H') = D^2 + n\underbrace{(D.H)}_{=0} > 0$$

$$\Rightarrow mD \text{ effective} \quad \text{for } m \gg 0$$

$$\Rightarrow (mD.H) = \deg_{\mathcal{O}_H}(mD) > 0 \quad ↯$$

Step 2 if $D^2 = 0$. $\quad D \not\sim_{num} 0 \quad \Rightarrow \exists\, E$ w/ $(D.E) \neq 0$

Replace E by $E' = H^2 E - (E.H)H$

$\Rightarrow$ wlog. $\quad E.H = 0$

$D' := nD + E \quad \Rightarrow \quad D'.H = 0, \quad D'^2 = 2n\underbrace{D.E}_{\neq 0} + E^2 > 0$ for $n \gg 0$

$\Rightarrow D'$ as in Step 1 $\lightning$

$\square$

Theorem (Nakai–Moishezon Criterion)

D divisor on X

D ample $\iff$ $D^2 > 0$ & $D.C > 0$ for $C \subseteq X$ irred.

Proof sketch "$\Rightarrow$" easy

mD v. ample for $m \gg 0$

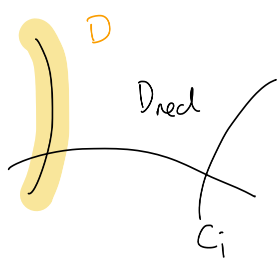
$\Rightarrow \quad m^2 D^2 > 0 \quad m(D.C) = \deg_{mD}(C) > 0$

"$\Leftarrow$"

H very ample repr. by ns. irred curve

$\Rightarrow \quad (D.H) > 0$

$\Rightarrow \quad mD$ effective for $m \gg 0$. replace D by $D' = mD$



$\mathcal{O}_X(D)|_D$ ample

$\Vert$ iff $\mathcal{O}_X(D)|_{C_{i,red}}$ ample.

sketchy $\Big\}$ $\deg_{C_i}(\mathcal{O}_X(D)|_{C_{i,red}}) = (D.C_i) > 0$

$\Rightarrow \mathcal{O}_X(D)|_D$ ample.

$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0$

LES

$$0 \to H^0(\mathcal{O}_X((n-1)D)) \to H^0(\mathcal{O}_X(nD)) \to H^0(\mathcal{O}_X(nD)|_D)$$
$$\to H^1(\mathcal{O}_X((n-1)D)) \to H^1(\mathcal{O}_X(nD)) \to \underbrace{H^1(\mathcal{O}_X(nD)|_D)}_{=0 \text{ for } n \gg 0}$$
$$\to \dots$$

$$\Rightarrow \infty > h^0(\mathcal{O}_X(nD)) \geq h^0(\mathcal{O}_X((n+1)D))$$
$$\geq h^0(\mathcal{O}_X((n+2)D)) \geq \cdots$$

$\Rightarrow$ stabilizes for $n \gg 0$

$\Rightarrow \quad H^0(\mathcal{O}_X(nD)) \twoheadrightarrow H^0(\mathcal{O}_X(nD)|_D)$ for $n \gg 0$

NAKAYAMA $\Rightarrow$ globally gen. sections of $\mathcal{O}_X(nD)|_D$ lift to gen sections @ pts of D

but $\mathcal{O}_X(nD)|_{X \setminus D} \cong \mathcal{O}_{X \setminus D}$

$\Rightarrow \quad \mathcal{O}_X(nD)$ globally generated everywhere.

$$\Rightarrow \quad \pi : X \longrightarrow \mathbb{P}(H^0(\mathcal{O}_X(nD))) = \mathbb{P}^N$$

Claim: π quasi finite. $\overset{\pi \text{ prj}}{\Longrightarrow}$ π finite.

Reason: if $\varphi(C) = pt$ $\quad C \subset X$ irred.

then $\exists$ hyperplane $E \subset \mathbb{P}^N$ s.t. $E \cap pt = \emptyset$

$\Rightarrow \quad (nD . C) = (\varphi^* E . C) = 0$ ↯

$\Rightarrow \quad \mathcal{O}_X(nD)$ ample. $\Rightarrow \mathcal{O}_X(D)$ ample.

pullbacks along finite surj. morph. are ample. $\square$.